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Homology rings of homotopy associative H-spaces

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ABSTRACT

Let X be a homotopy associative mod p H-space for p an odd prime. The homology $H_*(X;\mathbb{F}_p)$ is an associative ring, but not necessarily commutative. We study conditions when $[\bar{x},\bar{y}]\neq 0$ for $\bar{x},\bar{y}$ elements of $H_*(X;\mathbb{F}_p)$. Under certain conditions $[\bar{x},\bar{y}]\neq 0$ imply $ad^l(\bar{x},\bar{y})\neq 0$ for $l=p-2$ or $p-1$. These methods can be used to prove results about homology commutators that were previously obtained using the adjoint action [H. Hamanaka, S. Hara, A. Kono, Adjoint action of Lie groups on the loop spaces and cohomology of exceptional Lie groups, Transform. Group Theory (1996) 44–50, Korea Adv. Inst. Sci. Tech.; A. Kono, K. Kozima, The adjoint action of a Lie group on the space of loops, J. Math. Soc. Japan 45 (3) (1993) 495–509; A. Kono, J. Lin, O. Nishimura, Characterization of the mod 3 cohomology of E_7, Proc. Amer. Math. Soc. 131 (10) (2003) 3289–3295]. We also generalize results of Kane [R. Kane, Torsion in homotopy associative H-spaces, Illinois J. Math. 20 (1976) 476–485] to nonfinite mod p homotopy associative H-spaces.

1. Introduction

An H-space (X,μ) is a pointed space with basepoint preserving map $\mu: X\times X\to X$ where the basepoint acts as a two sided homotopy unit. X is homotopy associative if $(\mu\times 1)$ and $\mu(1\times\mu)$ are homotopic. For p a prime, a p local space is of finite type if $H_*(X;\mathbb{Z}_{(p)})$ is finitely generated in each degree.

In this paper we study the ring structure of $H_*(X;\mathbb{F}_p)$ where X is a homotopy associative p-local H-space of finite type for p an odd prime. Because X is homotopy associative, $H_*(X;\mathbb{F}_p)$ is an associative ring, but in general, it is not commutative.

Given $\bar{x},\bar{y}$ in $H_*(X;\mathbb{F}_p)$ the commutator $[\bar{y},\bar{x}]$ is defined by

$$[\bar{y},\bar{x}]=\bar{y}\bar{x}-(-1)^{|\bar{y}||\bar{x}|}\bar{x}\bar{y}$$

where $|\bar{x}|$ denotes the degree of $\bar{x}$. We define inductively

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$$ad^1(\bar{x})(\bar{y}) = [\bar{y}, \bar{x}],$$
$$ad^i(\bar{x})(\bar{y}) = \left[ad^{i-1}(\bar{x})(\bar{y}), \bar{x}\right].$$

We are led to the following Question 1.2 of [8] posed in 1995.

Question. Given X a homotopy associative p local H-space of finite type, when is $[\bar{x}, \bar{y}] \neq 0$ for $\bar{x}, \bar{y}$ in $H_*(X; \mathbb{F}_p)$?

It may be useful to provide some background about this question.

For homotopy associative H-spaces whose mod p cohomology is exterior on odd degree generators, the Samelson Leray theorem states that the homology is commutative. It is known that for finite H-spaces, there are no primitive homology commutators in even degrees [12]. The known examples of nontrivial homology commutators in finite simply connected mod p homotopy associative H-spaces occur in the exceptional Lie groups F_4, E_6, E_7 and E_8 for the primes 3 and 5 only [13]. One wonders why there are so few examples, given that one can construct many different homology Hopf algebras that are noncommutative.

In the case of these exceptional groups, we obtain commutators of the form $[\bar{x}, \bar{x}Q_{j+1}] \neq 0$ where $\bar{x}$ is of even degree and Q_{j+1} is the Milnor operation. Often one also has $ad^{p-2}(\bar{x})(\bar{x}Q_{j+1}) \neq 0$. Kane [4] is able to show this phenomenon occurs for finite homotopy associative H-spaces that contain even generators in their homology. The homotopy associativity is necessary as there are examples of finite H-spaces due to Harper [2] that are not homotopy associative with even homology generator $\bar{x}$, with $\bar{x}Q_1 \neq 0$ but no commutators $[\bar{x}, \bar{x}Q_1]$.

In this note we consider when $[\bar{x}, \bar{y}] \neq 0$ implies $ad^l(\bar{x})(\bar{y}) \neq 0$ for $l \leqslant p-1$, in the case when $\bar{y} \neq \bar{x}Q_{j+1}$.

We prove

Theorem 1 *(Main Theorem). Let X be a p-local homotopy associative H-space of finite type. Suppose the following conditions are satisfied*:

(a) *There is an element $\zeta \in H^*(X; \mathbb{F}_p)$ with $0 \neq \overline{\Delta}\zeta = x \otimes y \in PH^{2n}(X; \mathbb{F}_p) \otimes PH^{2m+1}(X; \mathbb{F}_p)$ with x indecomposable.*
(b) $\beta_1 \mathcal{P}^n \zeta = 0$.
(c) *x, y are generators of a Borel decomposition and $\bar{x}$, $\bar{y}$ are dual primitives in the coalgebra decomposition of $H_*(X; \mathbb{F}_p)$ with $\bar{x}^p = 0$ and $\langle \bar{x}, I(\mathcal{A}(p))y \rangle = 0 = \langle \bar{y}, I(\mathcal{A}(p))x \rangle$. Here $\mathcal{A}(p)$ is the* mod p *Steenrod algebra.*
(d)
$$\left(\bar{x}^{\otimes^{i-1}} \otimes \bar{y} \otimes \bar{x}^{\otimes p-i}\right)\beta_1 \mathcal{P}^n = 0,$$
$$\left(\bar{x}^{\otimes^{i-1}} \otimes \bar{y} \otimes \bar{x}^{\otimes p-i+1}\right)\beta_1 \mathcal{P}^n = 0$$
for $1 \leqslant i \leqslant p+1$.
Then
$$ad^{p-2}(\bar{x})(\bar{y}) \neq 0 \quad \textit{and if } m < n, \textit{ then}$$
$$ad^{p-1}(\bar{x})(\bar{y}) \neq 0.$$

Remark 2.

(a) Because $\overline{H}^*(X; \mathbb{F}_p)$ and $\overline{H}_*(X; \mathbb{F}_p)$ are dual Hopf algebras, $\langle [\bar{x}, \bar{y}], \zeta \rangle = \langle \bar{x} \otimes \bar{y} - (-1)^{|\bar{x}||\bar{y}|} \bar{y} \otimes \bar{x}, \overline{\Delta}\zeta \rangle$. Hence condition (a) of the Main Theorem implies $[\bar{x}, \bar{y}] \neq 0$.
(b) $H^*(X; \mathbb{F}_p)$ and $H_*(X; \mathbb{F}_p)$ are dual Hopf algebras over the Steenrod Algebra $\mathcal{A}(p)$. Let
$$\langle , \rangle : H_*(X; \mathbb{F}_p) \otimes H^*(X; \mathbb{F}_p) \to \mathbb{F}_p$$
denote the dual pairing. If $\bar{x}$ lies in $H_*(X; \mathbb{F}_p)$, y lies in $H^*(X; \mathbb{F}_p)$ and θ lies in $\mathcal{A}(p)$ then
$$\langle \bar{x}, \theta y \rangle = (-1)^{|\bar{x}||\theta|} \langle \bar{x}\theta, y \rangle.$$
Thus conditions (c) and (d) of the Main Theorem are statements about the action of the Steenrod algebra.

We now provide some examples of H-spaces, both finite and nonfinite, along with possible commutators in their homology rings. With these particular ring structures, we will show the H-structure on the H-space cannot be homotopy associative.

These examples are a small sample of the kinds of commutators that may be detected using this method. For example, using this technology, it is possible to obtain the commutators described by Hara, Hamanaka, Kono, Kozima, and Nishimura, [1,6,7] in their papers on the adjoint operation (see Theorem 13). We sketch a proof of this in Section 4.

Examples.

(1) We note $S^{2n+1}_{(p)}$ is a homotopy associative H-space for $p > 3$ [5].
Consider $X = E_8 \times S^{2m+1} \times S^{2m+13}$ localized at the prime 5 where E_8 is the exceptional Lie group. Assume that m is not 1 or 5. This has a homotopy associative mod 5 product H-structure.
Recall [13]

$$H^*(E_8; \mathbb{F}_5) = \Lambda(x_3, x_{11}, x_{15}, x_{23}, x_{27}, x_{35}, x_{39}, x_{47}) \otimes \frac{\mathbb{F}_5[x_{12}]}{x_{12}^5}.$$

Suppose there is an H-structure on X such that $\zeta \in H^{2m+13}(S^{2m+13}; \mathbb{F}_5)$ has $\overline{\Delta}\zeta = x_{12} \otimes y_{2m+1}$ for $y_{2m+1} \in H^{2m+1}(S^{2m+1}; \mathbb{F}_5)$. One can check the hypotheses (a)–(d) of the Main Theorem are satisfied (see Appendix A). Hence the Main Theorem would require additional generators in degrees $12l + 2m + 1$ for $l = 2, 3$ dual to $ad^l(\bar{x}_{12})(\bar{y}_{2m+1})$. We conclude X with this H-structure cannot be homotopy associative.

(2) Let E_6 be the exceptional Lie group. Recall [13]

$$H^*(E_6; \mathbb{F}_3) \cong \Lambda(x_3, x_7, x_9, x_{11}, x_{15}, x_{17}) \otimes \frac{\mathbb{F}_3[x_8]}{x_8^3}.$$

$x_9 \in H^9(E_6; \mathbb{F}_3)$ is primitive [12,13] and actually an A_3-class because its A_3 obstruction satisfies $A_3(x_9) \in H^8(E_6 \wedge E_6 \wedge E_6; \mathbb{F}_3) = 0$. Let $\widehat{E}_6$ be the fibre of the A_3-map

$$f : E_6 \to K(\mathbb{Z}_3, 9) \qquad f^*(i_9) = x_9.$$

Then $H^*(\widehat{E}_6; \mathbb{F}_3) \supseteq \Lambda(x_3, x_7, x_{11}, x_{15}, x_{17}) \otimes \mathbb{F}_3[x_8]/x_8^3$. $\widehat{E}_6$ is homotopy associative but is no longer a finite H-space.

Suppose X is an H-space whose cohomology satisfies $H^*(X; \mathbb{F}_3) \cong H^*(\widehat{E}_6; \mathbb{F}_3) \otimes \Lambda(y_3, \zeta_{11})$ with $\overline{\Delta}\zeta_{11} = x_8 \otimes y_3$.

Assume all Steenrod operations vanish on y_3. By the Main Theorem $ad^2(\bar{x}_8)(\bar{y}_3) \neq 0$ if X is homotopy associative. This would produce a 19-dimensional cohomology generator which does not exist in $H^*(X; \mathbb{F}_3)$. One can check the hypotheses (a)–(d) of the Main Theorem are satisfied (see Appendix A). We conclude X with this H-structure cannot be homotopy associative.

(3) Recall the mod 5 cohomology of the exceptional Lie group E_8 satisfies [13]

$$H^*(E_8; \mathbb{F}_5) = \frac{\mathbb{F}_5[x_{12}]}{x_{12}^5} \otimes \Lambda(x_3, x_{11}, x_{15}, x_{23}, x_{27}, x_{35}, x_{39}, x_{47})$$

with

$$\langle ad^l(\bar{x}_{12})(\bar{x}_{11}), x_{12l+11} \rangle \neq 0 \quad \text{for } l = 1, 2, 3.$$

Suppose X was an H-space with

$$H^*(X; \mathbb{F}_5) \cong H^*(E_8; \mathbb{F}_5) \otimes \Lambda(x_{51}, x_{62}, x_{75}, x_{87})$$

with $\mathcal{P}^5 x_{11} = x_{51}$. Then $\mathcal{P}^5 x_{23} = x_{63}$, $\mathcal{P}^5 x_{35} = x_{75}$, $\mathcal{P}^5 x_{47} = x_{87}$. These Steenrod connections are required because

$$ad^l(\bar{x}_{12})(\bar{x}_{51})\mathcal{P}^5 = ad^l(\bar{x}_{12})(\bar{x}_{11}) \neq 0 \quad \text{for } l = 1, 2, 3.$$

Such an H-space cannot be associative. Theorem 13 requires $ad^4(\bar{x}_{12})(\bar{x}_{51}) \neq 0$ which produces an additional generator in degree 99. Theorem 13 generalizes theorems of [1,6,7] on the adjoint action.

We describe here the ideas of the proof of these theorems. The reader may recall that for any given H-space X, the reduced coproduct $\overline{\Delta}$ is realizable by the Hopf construction map. The cofibre of this map is the projective plane P_2X. There is a cofibration sequence that relates the cohomology of X to the cohomology of the projective plane P_2X of X and the reduced coproduct.

The iterated reduced coproduct $\overline{\Delta}^{l-1}$ is also realizable by a map θ^{l-1} [9]. Taking cofibres, we obtain a cofibration sequence relating the cohomology of X to the cohomology of a different projective plane $\Gamma_l X$ and the iterated reduced coproduct. When the H-space is homotopy associative, for $l \geqslant 3$, we obtain homotopy commutative ladders relating $\Gamma_l X$ and $\Gamma_{l-1} X$. When one applies cohomology to this ladder, one obtains commutative ladders of exact sequences involving the iterated reduced coproducts (diagrams (2.2) and (2.3)).

The vanishing of certain Steenrod operations on elements of $H^*(X)$ yield various elements W in $\overline{H}^*(X)^{\otimes l}$. W may be considered to be a type of iterated H-deviation [3]. Commutativity of diagrams (2.2) and (2.3) imply that W may be expressed in two different ways, one involving $(1 \otimes \overline{\Delta}^{l-2})S_1$ and another involving $(\overline{\Delta}^{l-2} \otimes 1)S_2$ for elements $S_i \in \overline{H}^*(X)^{\otimes 2}$. Elements with certain nontrivial iterated reduced coproducts (see Eq. (2.5)) correspond dually to iterated commutators in the homology. The conditions of the Main Theorem are precisely the conditions needed to obtain the iterated commutators (see Eq. (2.8) and Propositions 11 and 16).

Throughout the paper we assume all spaces are connected and basepointed. All cohomology and homology groups will have coefficients in $\mathbb{F}_p$, the field of p elements for p an odd prime. If A is a Hopf algebra, $Q(A), P(A)$ will denote the modules of indecomposables and primitives respectively. We will use the term *generator* to denote an element whose projection into the module of indecomposables is nontrivial.

For a connected space Z, $\overline{H}^*(Z)$ will denote the augmentation ideal of elements of strictly positive degree.

If (Z,μ) is an H-space, $H^*(Z)$ becomes a Hopf algebra. The reduced coproduct $\overline{\Delta}: \overline{H}^*(Z) \to \overline{H}^*(Z) \otimes \overline{H}^*(Z)$ is defined in terms of the coproduct $\Delta: H^*(Z) \to H^*(Z) \otimes H^*(Z)$ as follows. Given an element $z \in \overline{H}^*(Z)$, $\overline{\Delta}(z) = \Delta(z) - z \otimes 1 - 1 \otimes z$. The reduced coproduct is dual to the multiplication induced by μ_* restricted to the ideal of positive degree elements of the homology $\overline{H}_*(Z) \otimes \overline{H}_*(Z) \to \overline{H}_*(Z)$.

We will reserve the symbol X to denote a homotopy associative p-local H-space of finite type. The symbol s will denote a generator of $H^1(S^1)$.

The symbol $\mathcal{A}(p)$ will denote the mod p Steenrod algebra, generated by symbols $\beta_1, \mathcal{P}^i$ for $i > 0$. The symbols Q_j are reserved for the Milnor operations [4] defined inductively by $Q_0 = \beta_1$, $Q_{j+1} = [\mathcal{P}^{p^j}, Q_j]$ for $j \geqslant 0$.

The symbol $T: Y \wedge Z \to Z \wedge Y$ is reserved for the map that interchanges the factors $T(y,z) = (z,y)$. Note that on cohomology

$$T^*: \overline{H}^*(Z) \otimes \overline{H}(Y) \to \overline{H}^*(Y) \otimes \overline{H}^*(Z) \quad \text{satisfies } T^*(a \otimes b) = (-1)^{|a||b|} b \otimes a.$$

2. Ladders related to homotopy associativity

Much of the material in this section is a summary of [9]. For $l \geqslant 2$ there is a cofibration sequence

$$S^1 \wedge X^{\wedge^l} \xrightarrow{\theta^{l-1}} S^1 \wedge X \xrightarrow{i_l} \Gamma_l X \xrightarrow{\lambda_l} S^1 \wedge S^1 \wedge X^{\wedge^l}.$$

Applying cohomology and identifying $(\theta^{l-1})^* = 1 \otimes \overline{\Delta}^{l-1}$ we obtain an exact sequence

$$\overline{H}^*(S^1) \otimes \overline{H}^*(X)^{\otimes^l} \xleftarrow{1\otimes\overline{\Delta}^{l-1}} \overline{H}^*(S^1) \otimes \overline{H}^*(X) \xleftarrow{i_l^*} H^*(\Gamma_l X) \xleftarrow{\lambda_l^*} \overline{H}^*(S^1)^{\otimes^2} \otimes \overline{H}^*(X)^{\otimes^l}. \tag{2.1}$$

If X is homotopy associative, then for $l \geqslant 3$, there exists a homotopy commutative ladder of cofibrations [9]

$$\begin{tikzcd}
X \wedge S^1 \wedge X^{\wedge^{l-1}} \arrow[r, "T\wedge 1"] \arrow[d, "1\wedge\theta^{l-2}"'] & S^1 \wedge X^{\wedge^l} \arrow[r, equal] \arrow[d, "\theta^{l-1}"] & S^1 \wedge X^{\wedge^l} \arrow[d, "\theta^{l-2}\wedge 1"] \\
X \wedge S^1 \wedge X \arrow[r, "\theta(T\wedge 1)"] \arrow[d, "1\wedge i_{l-1}"'] & S^1 \wedge X \arrow[d, "i_l"] & S^1 \wedge X^{\wedge^2} \arrow[l, "\theta"'] \arrow[d, "i_{l-1}\wedge 1"] \\
X \wedge \Gamma_{l-1} X \arrow[r, "\delta_1"] \arrow[d, "1\wedge\lambda_{l-1}"'] & \Gamma_l X \arrow[d, "\lambda_l"] & \Gamma_{l-1} X \wedge X \arrow[l, "\delta_2"'] \arrow[d, "\lambda_{l-1}\wedge 1"] \\
X \wedge S^1 \wedge S^1 \wedge X^{\wedge^{l-1}} \arrow[r, "T^2\wedge 1"'] & S^1 \wedge S^1 \wedge X^{\wedge^l} \arrow[r, equal] & S^1 \wedge S^1 \wedge X^{\wedge^l}
\end{tikzcd}$$

Applying cohomology we obtain the following commutative ladders where the columns are exact.

$$\begin{tikzcd}
\overline{H}^*(X) \otimes \overline{H}^*(S^1) \otimes \overline{H}^*(X)^{\otimes^{l-1}} & \overline{H}^*(S^1) \otimes \overline{H}^*(X)^{\otimes^l} \arrow[l, "T^*\otimes 1"'] \\
\overline{H}^*(X) \otimes \overline{H}^*(S^1) \otimes \overline{H}^*(X) \arrow[u, "1\otimes 1\otimes\overline{\Delta}^{l-2}"] & \overline{H}^*(S^1) \otimes \overline{H}^*(X) \arrow[l, "(T^*\otimes 1)(1\otimes\overline{\Delta})"'] \arrow[u, "1\otimes\overline{\Delta}^{l-1}"'] \\
\overline{H}^*(X) \otimes \overline{H}^*(\Gamma_{l-1} X) \arrow[u, "1\otimes i_{l-1}^*"] & H^*(\Gamma_l X) \arrow[l, "\delta_1^*"'] \arrow[u, "i_l^*"'] \\
\overline{H}^*(X) \otimes \overline{H}^*(S^1)^{\otimes^2} \otimes \overline{H}^*(X)^{\otimes^{l-1}} \arrow[u, "1\otimes\lambda_{l-1}^*"] & \overline{H}^*(S^1)^{\otimes^2} \otimes \overline{H}^*(X)^{\otimes^l} \arrow[l, "T^{2^*}\otimes 1"'] \arrow[u, "\lambda_l^*"']
\end{tikzcd} \tag{2.2}$$

$$\begin{tikzcd}
\overline{H}^*(S^1)\otimes\overline{H}^*(X)^{\otimes l} \arrow[r, equal] & \overline{H}^*(S^1)\otimes\overline{H}^*(X)^{\otimes l} \\
\overline{H}^*(S^1)\otimes\overline{H}^*(X) \arrow[u, "1\otimes\overline{\Delta}^{l-1}"] \arrow[r, "1\otimes\overline{\Delta}"] & \overline{H}^*(S^1)\otimes\overline{H}^*(X)^{\otimes 2} \arrow[u, "1\otimes\overline{\Delta}^{l-2}\otimes 1"'] \\
H^*(\Gamma_l X) \arrow[u, "i_l^*"] \arrow[r, "\delta_2^*"] & \overline{H}^*(\Gamma_{l-1}X)\otimes\overline{H}^*(X) \arrow[u, "i_{l-1}^*\otimes 1"'] \\
\overline{H}^*(S^1)^{\otimes 2}\otimes\overline{H}^*(X)^{\otimes l} \arrow[u, "\lambda_l^*"] \arrow[r, equal] & \overline{H}^*(S^1)^{\otimes 2}\otimes\overline{H}^*(X)^{\otimes l} \arrow[u, "\lambda_{l-1}^*\otimes 1"']
\end{tikzcd} \tag{2.3}$$

Theorem 3. *(See [9].) Let $z\in H^{2n}(X;\mathbb{F}_p)$ be an algebra generator with $\overline{\Delta}^{p-1}(z)=0$. Suppose $i_p^*(\hat{z})=s\otimes z$, for $\hat{z}\in H^{2n+1}(\Gamma_p X;\mathbb{F}_p)$. Then*

$$-\beta_1\mathcal{P}^n\hat{z}=\lambda_\rho^*(s\otimes s\otimes z\otimes\cdots\otimes z).$$

It will be useful to consider commutators in the tensor algebra of the homology of a space. Let $TIH_*(Z)$ be the tensor algebra of $IH_*(Z)$.

Definition 4. The external commutators in $TIH_*(Z)$ are defined by

$$\begin{aligned}
&Ad^1(\bar{x})(\bar{y})=[\![\bar{y},\bar{x}]\!]=\bar{y}\otimes\bar{x}-(-1)^{|\bar{y}||\bar{x}|}\bar{x}\otimes\bar{y},\\
&Ad^i(\bar{x})(\bar{y})=\left[\!\left[Ad^{i-1}(\bar{x})(\bar{y}),\bar{x}\right]\!\right]\quad\text{for } i>1.
\end{aligned}$$

By [4]

$$Ad^{p-1}(\bar{x})(\bar{y})=\sum_{i=1}^{p}\bar{x}^{\otimes^{i-1}}\otimes\bar{y}\otimes\bar{x}^{\otimes^{p-i}}. \tag{2.4}$$

If Z is an H-space, the reduced coproduct $\overline{\Delta}:\overline{H}^*(Z)\to\overline{H}^*(Z)^{\otimes^2}$ dualizes to multiplication in $\overline{H}_*(Z)$. Hence, we obtain the following relation between external and internal commutators [9]:

$$\begin{aligned}
&\left\langle[\![\bar{y},\bar{x}]\!],\overline{\Delta}v\right\rangle=\left\langle[\bar{y},\bar{x}],v\right\rangle\quad\text{and}\\
&\left\langle Ad^l(\bar{x})(\bar{y}),\overline{\Delta}^{l-1}v\right\rangle=\left\langle ad^l(\bar{x})(\bar{y}),v\right\rangle.
\end{aligned} \tag{2.5}$$

We now use diagrams (2.2) and (2.3) together with a diagram chase to prove

Proposition 5. *Given $y\in PH^{2m+1}(X)\cap\ker\beta_1\mathcal{P}^n$. Let $\hat{y}\in H^{2m+2}(\Gamma_p X)$ satisfy $i_p^*(\hat{y})=s\otimes y$ and $\beta_1\mathcal{P}^n\hat{y}=\lambda_p^*(s\otimes s\otimes W_1)$. Further, suppose $\langle Ad^{p-1}(\bar{x})(\bar{y}),W_1\rangle\neq 0$ and $(\bar{x}^{\otimes^{i-1}}\otimes\bar{y}\otimes\bar{x}^{\otimes^{p-i}})\beta_1\mathcal{P}^n=0$ for $1\leqslant i\leqslant p$. Then $ad^{p-2}(\bar{x})(\bar{y})\neq 0$.*

Proof. Let $l=p$ in diagrams (2.2) and (2.3). Then

$$\begin{aligned}
&\left(1\otimes i_{p-1}^*\right)\delta_1^*(\hat{y})=(T^*\otimes 1)(1\otimes\overline{\Delta})(s\otimes y)=0,\\
&\left(i_{p-1}^*\otimes 1\right)\delta_2^*(\hat{y})=(1\otimes\overline{\Delta})(s\otimes y)=0
\end{aligned}$$

because $\overline{\Delta}y=0$.

By (2.1), $\ker i_{p-1}^*=\operatorname{im}\lambda_{p-1}^*$. Therefore, there are $\alpha_i\in\overline{H}^*(X)^{\otimes^p}$ with

$$\begin{aligned}
&\delta_1^*(\hat{y})=\left(1\otimes\lambda_{p-1}^*\right)\left(T^{2^*}\otimes 1\right)(s\otimes s\otimes\alpha_1),\\
&\delta_2^*(\hat{y})=\left(\lambda_{p-1}^*\otimes 1\right)(s\otimes s\otimes\alpha_2).
\end{aligned} \tag{2.6}$$

Applying $\beta_1\mathcal{P}^n$ to (2.6) and using diagrams (2.2) and (2.3), we have

$$\begin{aligned}
\delta_1^*\left(\beta_1\mathcal{P}^n\hat{y}\right)&=\left(1\otimes\lambda_{p-1}^*\right)\left(T^{2^*}\otimes 1\right)\left(s\otimes s\otimes\beta_1\mathcal{P}^n\alpha_1\right)\\
&=\delta_1^*\left(\lambda_p^*(s\otimes s\otimes W_1)\right)\\
&=\left(1\otimes\lambda_{p-1}^*\right)\left(T^{2^*}\otimes 1\right)(s\otimes s\otimes W_1),
\end{aligned}$$

$$\begin{aligned}\delta_2^*\big(\beta_1\mathcal{P}^n\hat{y}\big) &= \big(\lambda_{p-1}^*\otimes 1\big)\big(s\otimes s\otimes \beta_1\mathcal{P}^n\alpha_2\big)\\ &= \delta_2^*\lambda_p^*(s\otimes s\otimes W_1)\\ &= \big(\lambda_{p-1}^*\otimes 1\big)(s\otimes s\otimes W_1).\end{aligned}\tag{2.7}$$

Since

$$\begin{aligned}&\ker 1\otimes\lambda_{p-1}^* = \operatorname{im} 1\otimes\overline{\Delta}^{p-2},\\ &\ker\lambda_{p-1}^*\otimes 1 = \operatorname{im} 1\otimes\overline{\Delta}^{p-2}\otimes 1.\end{aligned}$$

Eq. (2.7) implies there are $S_i\in\overline{H}^*(X)^{\otimes 2}$ such that

$$W_1 = \beta_1\mathcal{P}^n\alpha_1 + \big(1\otimes\overline{\Delta}^{p-2}\big)S_1 = \beta_1\mathcal{P}^n\alpha_2 + \big(\overline{\Delta}^{p-2}\otimes 1\big)S_2.\tag{2.8}$$

By (2.4) $Ad^{p-1}(\bar{x})(\bar{y}) = \sum_{i=1}^p \bar{x}^{\otimes^{i-1}}\otimes\bar{y}\otimes\bar{x}^{\otimes^{p-i}}$. By hypothesis $Ad^{p-1}(\bar{x})(\bar{y})\beta_1\mathcal{P}^n = 0$.

Then,

$$\begin{aligned}0 &\neq \big\langle Ad^{p-1}(\bar{x})(\bar{y}), W_1\big\rangle \quad\text{by hypothesis}\\ &= \big\langle Ad^{p-2}(\bar{x})(\bar{y})\otimes\bar{x} - \bar{x}\otimes Ad^{p-2}(\bar{x})(\bar{y}), W_1\big\rangle \quad\text{by Definition 4}\\ &= \big\langle Ad^{p-2}(\bar{x})(\bar{y})\otimes\bar{x}, W_1\big\rangle - \big\langle\bar{x}\otimes Ad^{p-2}(\bar{x})(\bar{y}), W_1\big\rangle\\ &= \big\langle Ad^{p-2}(\bar{x})(\bar{y})\otimes\bar{x}, \big(\overline{\Delta}^{p-2}\otimes 1\big)S_2\big\rangle - \big\langle\bar{x}\otimes Ad^{p-2}(\bar{x})(\bar{y}), \big(1\otimes\overline{\Delta}^{p-2}\big)S_1\big\rangle \quad\text{by (2.8)}\\ &= \big\langle ad^{p-2}(\bar{x})(\bar{y})\otimes\bar{x}, S_2\big\rangle - \big\langle\bar{x}\otimes ad^{p-2}(\bar{x})(\bar{y}), S_1\big\rangle \quad\text{by (2.5).}\end{aligned}$$

Therefore $ad^{p-2}(\bar{x})(\bar{y})\neq 0$. □

We can now generalize Kane's result to nonfinite H-spaces [4].

Proposition 6. *Let $y\in PH^{2m+1}(X)\cap\ker\beta_1\mathcal{P}^m$ and $x = Q_j y$ an indecomposable. Let $\bar{x}\in H_{2m+2p^j}(X)$ satisfy $\langle\bar{x},x\rangle = 1$ and $\bar{x}^p = 0$. Then*

$$ad^{p-2}(\bar{x})(\bar{x}Q_{j+1})\neq 0.$$

Proof. Let $\hat{y}\in H^{2m+2}(\Gamma_p X)$ satisfy $i_p^*(\hat{y}) = -s\otimes y$. Then $i_p^*(Q_j\hat{y}) = s\otimes x$. By Theorem 3,

$$-\beta_1\mathcal{P}^{m+p^j}Q_j\hat{y} = \lambda_p^*(s\otimes s\otimes x\otimes\cdots\otimes x).$$

By [4], $-\beta_1\mathcal{P}^{m+p^j}Q_j\hat{y} = Q_{j+1}\beta_1\mathcal{P}^m\hat{y}$. By hypothesis, $i_p^*(\beta_1\mathcal{P}^m\hat{y}) = s\otimes\beta_1\mathcal{P}^m y = 0$. Therefore by (2.1), $\beta_1\mathcal{P}^m\hat{y} = \lambda_p^*(s\otimes s\otimes W_1)$ for some W_1.

This implies

$$\begin{aligned}-\beta_1\mathcal{P}^{m+p^j}Q_j\hat{y} = Q_{j+1}\beta_1\mathcal{P}^m\hat{y} &= \lambda_p^*(s\otimes s\otimes Q_{j+1}W_1)\\ &= \lambda_p^*(s\otimes s\otimes x\otimes\cdots\otimes x).\end{aligned}$$

By (2.1), $\ker\lambda_p^* = \operatorname{im} 1\otimes\overline{\Delta}^{p-1}$ implies there is a $\gamma\in\overline{H}^*(X)$ with $Q_{j+1}W_1 = x\otimes\cdots\otimes x + \overline{\Delta}^{p-1}\gamma$.

Then

$$\begin{aligned}\big\langle\bar{x}\otimes\cdots\otimes\bar{x}, x\otimes\cdots\otimes x + \overline{\Delta}^{p-1}\gamma\big\rangle &= \langle\bar{x},x\rangle^p + \big\langle\bar{x}^p,\gamma\big\rangle\\ &= 1 \quad\text{since } \bar{x}^p = 0\\ &= \langle\bar{x}\otimes\cdots\otimes\bar{x}, Q_{j+1}W_1\rangle\\ &= \big\langle(\bar{x}\otimes\cdots\otimes\bar{x})Q_{j+1}, W_1\big\rangle = \Big\langle\sum\bar{x}^{\otimes^{i-1}}\otimes\bar{x}Q_{j+1}\otimes\bar{x}^{\otimes^{p-i}}, W_1\Big\rangle\\ &= \big\langle Ad^{p-1}(\bar{x})(\bar{x}Q_{j+1}), W_1\big\rangle \quad\text{by (2.4).}\end{aligned}$$

We note that Proposition 5 applies with $\bar{y} = \bar{x}Q_{j+1}$. The degree of $\bar{x}^{\otimes^{i-1}}\otimes\bar{x}Q_{j+1}\otimes\bar{x}^{\otimes^{p-i}}$ is $2mp+1$. Thus $(\bar{x}^{\otimes^{i-1}}\otimes\bar{x}Q_{j+1}\otimes\bar{x}^{\otimes^{p-i}})\beta_1\mathcal{P}^m = 0$. Hence

$$ad^{p-2}(\bar{x})(\bar{x}Q_{j+1})\neq 0. \qquad\square$$

3. Proof of the Main Theorem

The proof comes from a diagram chase using diagrams (2.2) and (2.3) for $l = p + 1$.

By assumption of the Main Theorem

$$\overline{\Delta}\zeta = x \otimes y \in PH^{2n}(X) \otimes PH^{2m+1}(X).$$

Hence $\overline{\Delta}^p \zeta = 0$. By (2.1), there exists a $\hat{\zeta} \in H^*(\Gamma_{p+1}X)$ with $i^*_{p+1}(\hat{\zeta}) = s \otimes \zeta$. Further $\overline{\Delta}^{p-1}x = \overline{\Delta}^{p-1}y = 0$ so there exist elements $\hat{x}, \hat{y}$ in $H^*(\Gamma_p X)$ with $i^*_p(\hat{x}) = s \otimes x$ and $i^*_p(\hat{y}) = s \otimes y$.

By diagram (2.2),

$$\begin{aligned}(1 \otimes i^*_p)\delta^*_1(\hat{\zeta}) &= (T^* \otimes 1)(1 \otimes \overline{\Delta})(s \otimes \zeta)\\ &= x \otimes s \otimes y = (1 \otimes i^*_p)(x \otimes \hat{y}).\end{aligned}$$

By diagram (2.3),

$$\begin{aligned}(i^*_p \otimes 1)\delta^*_2(\hat{\zeta}) &= (1 \otimes \overline{\Delta})(s \otimes \zeta)\\ &= s \otimes x \otimes y = (i^*_p \otimes 1)(\hat{x} \otimes y).\end{aligned}$$

By exactness of the columns of diagrams (2.2) and (2.3),

$$\ker i^*_p \otimes 1 = \operatorname{im} \lambda^*_p \otimes 1, \qquad \ker 1 \otimes i^*_p = \operatorname{im} 1 \otimes \lambda^*_p = \operatorname{im}(1 \otimes \lambda^*_p)(T^{2^*} \otimes 1).$$

Hence there exist elements $\alpha_i \in \overline{H}^*(X)^{\otimes^{p+1}}$ with

$$\begin{aligned}\delta^*_1(\hat{\zeta}) &= x \otimes \hat{y} + (1 \otimes \lambda^*_p)(T^{2^*} \otimes 1)(s \otimes s \otimes \alpha_1),\\ \delta^*_2(\hat{\zeta}) &= \hat{x} \otimes y + (\lambda^*_p \otimes 1)(s \otimes s \otimes \alpha_2).\end{aligned} \tag{3.1}$$

By hypothesis $-s \otimes \beta_1 \mathcal{P}^n \zeta = 0 = i^*_{p+1}(\beta_1 \mathcal{P}^n \hat{\zeta})$. Hence,

$$\beta_1 \mathcal{P}^n \hat{\zeta} = \lambda^*_{p+1}(s \otimes s \otimes W) \quad \text{for some } W \in \overline{H}^*(X)^{\otimes^{p+1}}.$$

Diagrams (2.2) and (2.3) imply

$$\begin{aligned}\delta^*_1(\beta_1 \mathcal{P}^n \hat{\zeta}) &= \delta^*_1 \lambda^*_{p+1}(s \otimes s \otimes W)\\ &= (1 \otimes \lambda^*_p)(T^{2^*} \otimes 1)(s \otimes s \otimes W),\\ \delta^*_2(\beta_1 \mathcal{P}^n \hat{\zeta}) &= \delta^*_2 \lambda^*_{p+1}(s \otimes s \otimes W)\\ &= (\lambda^*_p \otimes 1)(s \otimes s \otimes W).\end{aligned} \tag{3.2}$$

By (3.1) and the Cartan formula, we have an alternative formula for $\delta^*_i(\beta_1 \mathcal{P}^n \hat{\zeta})$

$$\begin{aligned}\delta^*_1(\beta_1 \mathcal{P}^n \hat{\zeta}) &= x \otimes \beta_1 \mathcal{P}^n \hat{y} + \sum_{k=0}^{n} \beta_1 \mathcal{P}^k x \otimes \mathcal{P}^{n-k} \hat{y} + \sum_{k=1}^{n} \mathcal{P}^k x \otimes \beta_1 \mathcal{P}^{n-k} \hat{y}\\ &\quad + (1 \otimes \lambda^*_p)(T^{2^*} \otimes 1)(s \otimes s \otimes \beta_1 \mathcal{P}^n \alpha_1),\end{aligned} \tag{3.3}$$

$$\begin{aligned}\delta^*_2(\beta_1 \mathcal{P}^n \hat{\zeta}) &= \sum_{k=0}^{n-1} \beta_1 \mathcal{P}^k \hat{x} \otimes \mathcal{P}^{n-k} y + \beta_1 \mathcal{P}^n \hat{x} \otimes y - \sum_{k=0}^{n} \mathcal{P}^k \hat{x} \otimes \beta_1 \mathcal{P}^{n-k} y\\ &\quad + (\lambda^*_p \otimes 1)(s \otimes s \otimes \beta_1 \mathcal{P}^n \alpha_2).\end{aligned} \tag{3.4}$$

Now $\beta_1 \mathcal{P}^n \zeta = 0$ implies $\beta_1 \mathcal{P}^n \overline{\Delta}\zeta = \beta_1 \mathcal{P}^n(x \otimes y) = 0$. Hence in $\beta_1 \mathcal{P}^k x \otimes \mathcal{P}^{n-k} y$ and $\mathcal{P}^k x \otimes \beta_1 \mathcal{P}^{n-k} y$ one of the factors is trivial. Therefore,

$$\begin{aligned}&(i^*_p \otimes 1)(\beta_1 \mathcal{P}^k \hat{x} \otimes \mathcal{P}^{n-k} y) = 0,\\ &(i^*_p \otimes 1)(\mathcal{P}^k \hat{x} \otimes \beta_1 \mathcal{P}^{n-k} y) = 0,\\ &(1 \otimes i^*_p)(x \otimes \beta_1 \mathcal{P}^n \hat{y}) = 0,\\ &(1 \otimes i^*_p)(\beta_1 \mathcal{P}^k x \otimes \mathcal{P}^{n-k} \hat{y}) = 0,\\ &(1 \otimes i^*_p)(\mathcal{P}^k x \otimes \beta_1 \mathcal{P}^{n-k} \hat{y}) = 0.\end{aligned}$$

Hence,

$$\begin{aligned}
&\beta_1\mathcal{P}^k\hat{x}\otimes\mathcal{P}^{n-k}y\in\left(\lambda_p^*\otimes 1\right)\left(s\otimes s\otimes\overline{H}^*(X)^{\otimes^p}\otimes\mathcal{P}^{n-k}y\right),\\
&\mathcal{P}^k\hat{x}\otimes\beta_1\mathcal{P}^{n-k}y\in\left(\lambda_p^*\otimes 1\right)\left(s\otimes s\otimes\overline{H}^*(X)^{\otimes^p}\otimes\beta_1\mathcal{P}^{n-k}y\right),\\
&\beta_1\mathcal{P}^n\hat{x}\otimes y=-\left(\lambda_p^*\otimes 1\right)(s\otimes s\otimes x\otimes\cdots\otimes x\otimes y)\quad\text{by Theorem 3},\\
&\beta_1\mathcal{P}^k x\otimes\mathcal{P}^{n-k}\hat{y}\in\left(1\otimes\lambda_p^*\right)\left(T^{2^*}\otimes 1\right)\left(s\otimes s\otimes\beta_1\mathcal{P}^k x\otimes\overline{H}^*(X)^{\otimes^p}\right),\\
&\mathcal{P}^k x\otimes\beta_1\mathcal{P}^{n-k}\hat{y}\in\left(1\otimes\lambda_p^*\right)\left(T^{2^*}\otimes 1\right)\left(s\otimes s\otimes\mathcal{P}^k x\otimes\overline{H}^*(X)^{\otimes^p}\right).
\end{aligned}\tag{3.5}$$

If

$$\begin{aligned}
&\beta_1\mathcal{P}^n\hat{y}=\lambda_p^*(s\otimes s\otimes W_1)\quad\text{then}\\
&x\otimes\beta_1\mathcal{P}^n\hat{y}=\left(1\otimes\lambda_p^*\right)\left(T^{2^*}\otimes 1\right)(s\otimes s\otimes x\otimes W_1).
\end{aligned}$$

By exactness of the cofibration sequences in diagrams (2.2) and (2.3),

$$\ker\lambda_p^*\otimes 1=\operatorname{im}1\otimes\overline{\Delta}^{p-1}\otimes 1\qquad\ker\left(1\otimes\lambda_p^*\right)\left(T^{2^*}\otimes 1\right)=\operatorname{im}1\otimes\overline{\Delta}^{p-1}.$$

Eqs. (3.2)–(3.5) imply

Theorem 7.

(a) *$W\in x\otimes W_1+\sum_{k=0}^n\beta_1\mathcal{P}^k x\otimes\overline{H}^*(X)^{\otimes^p}+\sum_{k=1}^n\mathcal{P}^k x\otimes\overline{H}^*(X)^{\otimes^p}+\beta_1\mathcal{P}^n\alpha_1+(1\otimes\overline{\Delta}^{p-1})S_2$ for some $S_2\in\overline{H}^*(X)^{\otimes^2}$ where $\lambda_p^*(s\otimes s\otimes W_1)=\beta_1\mathcal{P}^n\hat{y}$.*

(b)

$$\begin{aligned}
W\in{}&\sum_{k=0}^{n-1}\overline{H}^*(X)^{\otimes^p}\otimes\mathcal{P}^{n-k}y-x\otimes\cdots\otimes x\otimes y+\sum_{k=0}^{n}\overline{H}^*(X)^{\otimes^p}\otimes\beta_1\mathcal{P}^{n-k}y\\
&+\beta_1\mathcal{P}^n\alpha_2+\left(\overline{\Delta}^{p-1}\otimes 1\right)S_1
\end{aligned}$$

for some $S_1\in\overline{H}^*(X)^{\otimes^2}$.

Proof. We prove part (b).

Let

$$B=\left[\sum_{k=0}^{n-1}\overline{H}^*(X)^{\otimes^p}\otimes\mathcal{P}^{n-k}y-x\otimes\cdots\otimes x\otimes y+\sum_{k=0}^{n}\overline{H}^*(X)^{\otimes^p}\otimes\beta_1\mathcal{P}^{n-k}y+\beta_1\mathcal{P}^n\alpha_2\right].$$

Then by (3.2), (3.4), and (3.5), $(\lambda_p^*\otimes 1)(s\otimes s\otimes W)\in(\lambda_p^*\otimes 1)(s\otimes s\otimes B)$. Since $\ker\lambda_p^*\otimes 1=\operatorname{im}1\otimes 1\otimes\overline{\Delta}^{p-1}\otimes 1$ the result follows. The proof of part (a) is similar. □

Definition 8. Let $\Omega=\operatorname{Span}\{x^{\otimes^{i-1}}\otimes y\otimes x^{\otimes^{p-i+1}}:\ 1\leqslant i\leqslant p+1\}$.

We can now describe the strategy of the proof of the Main Theorem. By hypothesis of the Main Theorem, x and y are generators of a Borel decomposition for $H^*(X)$ and $\bar{x},\bar{y}$ are dual primitives in a Borel coalgebra decomposition of $H_*(X)$. The Borel decomposition provides us with a basis for $H^*(X)$ consisting of monomials in the Borel generators. To check if $ad^l(\bar{x})(\bar{y})$ is nontrivial we have by (2.5)

$$\left\langle ad^l(\bar{x})(\bar{y}),v\right\rangle=\left\langle Ad^l(\bar{x})(\bar{y}),\overline{\Delta}^{l-1}v\right\rangle.$$

Hence it suffices to check only the component of $\overline{\Delta}^{l-1}v$ in

$$\operatorname{Span}\left\{x^{\otimes^{i-1}}\otimes y\otimes x^{\otimes^{l-i+1}}:\ 1\leqslant i\leqslant l+1\right\}.$$

Theorem 7 provides us with two formulas for W. The two formulas for W involve terms $(1\otimes\overline{\Delta}^{p-1})S_2$ and $(\overline{\Delta}^{p-1}\otimes 1)S_1$. We can then calculate the component of W in Ω. By assumptions of the Main Theorem, the Steenrod modules $I(\mathcal{A}(p))x$ annihilates $\bar{y}$ and $I(\mathcal{A}(p))y$ annihilates $\bar{x}$. Further,

$$\left(\bar{x}^{\otimes^{i-1}}\otimes\bar{y}\otimes\bar{x}^{\otimes^{p-i+1}}\right)\beta_1\mathcal{P}^n=0.$$

Hence, almost all of the terms appearing in Theorem 7 have no component in Ω.

Lemma 9. *Terms of the form*

(a) $\beta_1\mathcal{P}^k x \otimes \overline{H}^*(X)^{\otimes^p}$,
(b) $\mathcal{P}^k x \otimes \overline{H}^*(X)^{\otimes^p}$, $k > 0$,
(c) $\overline{H}^*(X)^{\otimes^p} \otimes \mathcal{P}^{n-k} y$, $k < n$,
(d) $\overline{H}^*(X)^{\otimes^p} \otimes \beta_1\mathcal{P}^{n-k} y$,
(e) $\beta_1\mathcal{P}^n\alpha_i$

have no component in Ω.

Definition 10. $U, V \in \overline{H}^*(X)^{\otimes^{p+1}}$ are equivalent, written $U \sim V$ if they have the same component in Ω.

Proposition 11.

$$\begin{aligned} W &\sim x \otimes W_1 + \big(1 \otimes \overline{\Delta}^{p-1}\big) S_2 \\ &\sim -x \otimes \cdots \otimes x \otimes y + \big(\overline{\Delta}^{p-1} \otimes 1\big) S_1. \end{aligned}$$

Proof. This follows from Theorem 7 and Lemma 9. $\square$

Lemma 12. *Let $\sum_{i=1}^{p+1} \gamma_i x^{\otimes^{i-1}} \otimes y \otimes x^{\otimes^{p-i+1}}$ be the component of $(\overline{\Delta}^{p-1} \otimes 1)S_1$ in Ω. Then*

(a) $\gamma_1 = \gamma_{p+1} = 0$.

(b) $$\sum_{i=2}^{p} \gamma_i = \big\langle Ad^{p-1}(\bar{x})(\bar{y}) \otimes \bar{x}, \big(\overline{\Delta}^{p-1} \otimes 1\big) S_1\big\rangle = \big\langle ad^{p-1}(\bar{x})(\bar{y}) \otimes \bar{x}, S_1\big\rangle.$$

Proof.

$$\begin{aligned} \gamma_{p+1} &= \left\langle \bar{x} \otimes \cdots \otimes \bar{x} \otimes \bar{y}, \sum \gamma_i x^{\otimes^{i-1}} \otimes y \otimes x^{\otimes^{p-i+1}} \right\rangle \\ &= \big\langle \bar{x} \otimes \cdots \otimes \bar{x} \otimes \bar{y}, \big(\overline{\Delta}^{p-1} \otimes 1\big) S_1\big\rangle \\ &= \big\langle \bar{x}^p \otimes \bar{y}, S_1\big\rangle \\ &= 0 \quad \text{because } \bar{x}^p = 0 \text{ by hypothesis.} \end{aligned}$$

By Proposition 11,

$$\big(1 \otimes \overline{\Delta}^{p-1}\big) S_2 \sim -x \otimes W_1 - x \otimes \cdots \otimes x \otimes y + \big(\overline{\Delta}^{p-1} \otimes 1\big) S_1.$$

If $\gamma_1 \neq 0$, then $(1 \otimes \overline{\Delta}^{p-1})S_2$ has $\gamma_1 y \otimes x \otimes \cdots \otimes x$ as a nonzero summand. This would imply $\bar{x}^p \neq 0$.
Hence, $\gamma_1 = 0$.
Now

$$\begin{aligned} \sum_{i=2}^{p} \gamma_i = \sum_{i=1}^{p} \gamma_i &= \left\langle \sum_{i=1}^{p} \bar{x}^{\otimes^{i-1}} \otimes \bar{y} \otimes \bar{x}^{\otimes^{p-i}} \otimes \bar{x}, \big(\overline{\Delta}^{p-1} \otimes 1\big) S_1 \right\rangle \\ &= \big\langle Ad^{p-1}(\bar{x})(\bar{y}) \otimes \bar{x}, \big(\overline{\Delta}^{p-1} \otimes 1\big) S_1\big\rangle \quad \text{by (2.4)} \\ &= \big\langle ad^{p-1}(\bar{x})(\bar{y}) \otimes \bar{x}, S_1\big\rangle. \qquad \square \end{aligned}$$

We now prove the Main Theorem.

Case 1. $\langle Ad^{p-1}(\bar{x})(\bar{y}), W_1\rangle = 0$. We note if $m < n$ then $\beta_1\mathcal{P}^n\hat{y} = 0$ because deg $\hat{y} = 2m + 2$ and $m + 1 \leqslant n$. Hence W_1 may be chosen to be trivial.

By Proposition 11,

$$\begin{aligned} &\big\langle \bar{x} \otimes Ad^{p-1}(\bar{x})(\bar{y}), W\big\rangle \\ &\quad = \big\langle \bar{x} \otimes Ad^{p-1}(\bar{x})(\bar{y}), x \otimes W_1 + \big(1 \otimes \overline{\Delta}^{p-1}\big) S_2\big\rangle \\ &\quad = \big\langle \bar{x} \otimes ad^{p-1}(\bar{x})(\bar{y}), S_2\big\rangle \\ &\quad = \big\langle \bar{x} \otimes Ad^{p-1}(\bar{x})(\bar{y}), -x \otimes \cdots \otimes x \otimes y + \big(\overline{\Delta}^{p-1} \otimes 1\big) S_1\big\rangle \quad \text{by Proposition 11} \end{aligned}$$

$$= -1 + \left\langle \bar{x} \otimes \sum_{i=1}^{p} \bar{x}^{\otimes^{i-1}} \otimes \bar{y} \otimes \bar{x}^{\otimes^{p-i}}, \sum_{i=2}^{p} \gamma_i x^{\otimes^{i-1}} \otimes y \otimes x^{\otimes^{p-i+1}} \right\rangle \quad \text{by Lemma 12 and (2.4)}$$

$$= -1 + \sum_{i=2}^{p} \gamma_i.$$

If $-1 + \sum_{i=2}^{p} \gamma_i \neq 0$ then $ad^{p-1}(\bar{x})(\bar{y}) \neq 0$.
If $-1 + \sum_{i=2}^{p} \gamma_i = 0$ then $\sum_{i=2}^{p} \gamma_i = 1 \neq 0$.
By Lemma 12(b), $ad^{p-1}(\bar{x})(\bar{y}) \neq 0$.

Case 2. $\langle Ad^{p-1}(\bar{x})(\bar{y}), W_1 \rangle \neq 0$.

By Proposition 5, $ad^{p-2}(\bar{x})(\bar{y}) \neq 0$.

4. Commutators and the adjoint action

The adjoint action is used to detect homology commutators. We give a brief summary without proofs of the use of the adjoint action here. Details can be found in [1,6,7].

If X is a homotopy associative H-space, it has a homotopy inverse. The adjoint action is the map

$$\mathrm{Ad} : X \times \Omega X \to \Omega X$$

defined by $\mathrm{Ad}(g, \lambda) = g\lambda g^{-1}$. This induces a bilinear map

$$\mathrm{Ad}_* : H_*(X) \otimes H_*(\Omega X) \to H_*(\Omega X).$$

Given elements $\bar{x} \in H_*(X)$ and $\tilde{y} \in H_*(\Omega X)$ we define $\mathrm{Ad}_*(\bar{x} \otimes \tilde{y}) = \bar{x} * \tilde{y}$. The relation between the adjoint action and commutators in $H_*(X)$ can be summarized by the following fact. If $\bar{x}$ is primitive, and $\sigma_* : H_*(\Omega X) \to H_*(X)$ is the homology suspension, then

$$\sigma_*(\bar{x} * \tilde{y}) = \left[\bar{x}, \sigma_*(\tilde{y})\right].$$

Thus, information about the adjoint action can produce information about homology commutators, especially in those degrees where σ_* is monic.

The following calculation occurs frequently in [1,6,7]. Let $x \in H^{2n}(X)$, $y \in H^{2m+1}(X)$, $\mathcal{P}^m y$ be generators of a Borel decomposition of $H^*(X)$. Let $\bar{x} \in PH_{2n}(X)$ and $\bar{y} \in PH_{2m+1}(X)$, $\overline{\mathcal{P}^m y} \in PH_{2mp+1}(X)$ be part of a dual Borel coalgebra decomposition of $H_*(X)$.

Let $\sigma_*(\tilde{y}) = \bar{y}$, $\sigma_*(\gamma(\tilde{y})) = \overline{\mathcal{P}^m y}$ for $\gamma(\tilde{y}) \in H_{2mp}(X)$. Then

$$0 \neq \left\langle \sigma_*\left(\gamma(\tilde{y})\right), \mathcal{P}^m y \right\rangle = \left\langle \gamma(\tilde{y}), \sigma^*\left(\mathcal{P}^m y\right)\right\rangle = \left\langle \gamma(\tilde{y}), \sigma^*(y)^p \right\rangle.$$

The element $\gamma(\tilde{y})$ may be chosen so that

$$\overline{\Delta}\gamma(\tilde{y}) = \sum_{i=1}^{p-1} (i, p-i) \frac{1}{p} \tilde{y}^i \otimes \tilde{y}^{p-i}.$$

Suppose $0 \neq [\bar{x}, \bar{y}] = \sigma_*(\bar{x} * \tilde{y})$. Further suppose $\bar{x}^p = 0$ and $(\bar{x} * \tilde{y})^p \neq 0$. Then $\bar{x} * \gamma(\tilde{y}) = v - \tilde{y}^{p-1}(\bar{x} * \tilde{y})$ for some nonzero $v \in PH_{2n+2mp}(\Omega X)$. Further,

$$\bar{x}^{p-1} * \left(\bar{x} * \gamma(\tilde{y})\right) = \bar{x}^p * \gamma(\tilde{y}) = 0 = \bar{x}^{p-1} * v - (\bar{x} * \tilde{y})^p.$$

Hence

$$\bar{x}^{p-1} * v = (\bar{x} * \tilde{y})^p \neq 0.$$

Then

$$\bar{x}^{p-2} * v \neq 0$$

and

$$\sigma_*\left(\bar{x}^{p-2} * v\right) = ad^{p-1}(\bar{x})\left(\overline{\mathcal{P}^m y}\right).$$

If σ_* is monic in degree $2n(p-1) + 2mp$, then $ad^{p-1}(\bar{x})(\overline{\mathcal{P}^m y}) \neq 0$.

Let $\zeta \in H^{2n+2mp+1}(X)$ satisfy $\overline{\Delta}\zeta = x \otimes y$. Using universal examples one can show that if $\beta_1 \mathcal{P}^{n+m} \zeta = 0$, then $(\bar{x} \otimes \tilde{y})^p \neq 0$.

In this section we will prove a result that is similar to the result obtained above using the adjoint action.

Theorem 13. *Let X be a p-local homotopy associative H-space of finite type. Suppose the following conditions are satisfied*:

(a) *There is an element $\zeta \in H^*(X;\mathbb{F}_p)$ with $0 \neq \overline{\Delta}\zeta = x \otimes y \in PH^{2n}(X;\mathbb{F}_p) \otimes PH^{2m+1}(X;\mathbb{F}_p)$ with x indecomposable.*
(b) *$\beta_1\mathcal{P}^{n+m}\zeta = 0$.*
(c) *x, y and $\mathcal{P}^m y$ are generators of a Borel decomposition and $\bar{x}$, $\bar{y}$, $\overline{\mathcal{P}^m y}$ are dual primitives in the coalgebra decomposition of $H_*(X;\mathbb{F}_p)$ with $\bar{x}^p = 0$ and $\langle \bar{x}, \beta_1\mathcal{P}^m y\rangle = 0$.*
Then

$$ad^{p-1}(\bar{x})\big(\overline{\mathcal{P}^m y}\big) \neq 0.$$

By hypothesis (a) there exist elements $\hat{\zeta} \in H^*(\Gamma_{p+1}X)$, $\hat{x}$, $\hat{y} \in H^*(\Gamma_p X)$ with

$$i_{p+1}^*(\hat{\zeta}) = s \otimes \zeta, \qquad i_p^*(\hat{x}) = s \otimes x, \qquad i_p^*(\hat{y}) = s \otimes y \tag{4.1}$$

$\beta_1\mathcal{P}^{n+m}\zeta = 0$ implies $0 = \beta_1\mathcal{P}^{n+m}(x \otimes y) = x^p \otimes \beta_1\mathcal{P}^m y$, so either $x^p = 0$ or $\beta_1\mathcal{P}^m y = 0$. We have

$$\beta_1\mathcal{P}^{n+m}\hat{\zeta} = \lambda_{p+1}^*(s \otimes s \otimes W) \tag{4.2}$$

and by diagrams (2.2) and (2.3) for $l = p+1$

$$\begin{aligned}&\big(1 \otimes i_p^*\big)\delta_1^*(\hat{\zeta}) = x \otimes s \otimes y = \big(1 \otimes i_p^*\big)(x \otimes \hat{y}),\\&\big(i_p^* \otimes 1\big)\delta_2^*(\hat{\zeta}) = s \otimes x \otimes y = \big(i_p^* \otimes 1\big)(\hat{x} \otimes y).\end{aligned} \tag{4.3}$$

Therefore there exist $\alpha_i \in \overline{H}(X)^{\otimes p}$ with $|\alpha_i| = 2n + 2m$ with

$$\begin{aligned}&\delta_1^*(\hat{\zeta}) = x \otimes \hat{y} + \big(1 + \lambda_p^*\big)\big(T^{2^*} \otimes 1\big)(s \otimes s \otimes \alpha_1),\\&\delta_2^*(\hat{\zeta}) = \hat{x} \otimes y + \big(\lambda_p^* \otimes 1\big)(s \otimes s \otimes \alpha_2).\end{aligned} \tag{4.4}$$

Applying $\beta_1\mathcal{P}^{n+m}$ we obtain

$$\begin{aligned}&\delta_1^*\big(\beta_1\mathcal{P}^{n+m}\hat{\zeta}\big) = x^p \otimes \beta_1\mathcal{P}^m\hat{y},\\&\delta_2^*\big(\beta_1\mathcal{P}^{n+m}\hat{\zeta}\big) = \beta_1\mathcal{P}^n\hat{x} \otimes \mathcal{P}^m y - \mathcal{P}^n\hat{x} \otimes \beta_1\mathcal{P}^m y.\end{aligned} \tag{4.5}$$

Because $x^p = 0$ or $\beta_1\mathcal{P}^m y = 0$ there exist elements $W_i \in \overline{H}^*(X)^{\otimes p}$ with

$$\begin{aligned}&x^p \otimes \beta_1\mathcal{P}^m\hat{y} = \big(1 \otimes \lambda_p^*\big)\big(T^{2^*} \otimes 1\big)\big(s \otimes s \otimes x^p \otimes W_1\big),\\&\beta_1\mathcal{P}^n\hat{x} \otimes \mathcal{P}^m y = \big(\lambda_p^* \otimes 1\big)\big(s \otimes s \otimes -x \otimes \cdots \otimes x \otimes \mathcal{P}^m y\big) \quad \text{by Theorem 3},\\&-\mathcal{P}^n\hat{x} \otimes \beta_1\mathcal{P}^m y = \big(\lambda_p^* \otimes 1\big)\big(s \otimes s \otimes W_2 \otimes \beta_1\mathcal{P}^m y\big).\end{aligned}$$

Proposition 14.

(a) *$W = x^p \otimes W_1 + (1 \otimes \overline{\Delta}^{p-1})S_2$ for some $S_2 \in \overline{H}(X)^{\otimes 2}$.*
(b) *$W = -x \otimes \cdots \otimes x \otimes \mathcal{P}^m y - W_2 \otimes \beta_1\mathcal{P}^m y + (\overline{\Delta}^{p-1} \otimes 1)S_1$ for some $S_1 \in \overline{H}(X)^{\otimes 2}$.*

Proof. We prove part (b). We have

$$\begin{aligned}&\big(\lambda_p^* \otimes 1\big)\big(s \otimes s \otimes -x \otimes \cdots \otimes x \otimes \mathcal{P}^m y - W_2 \otimes \beta_1\mathcal{P}^m y\big)\\&\quad = \delta_2^*\big(\beta_1\mathcal{P}^{n+m}\hat{\zeta}\big)\\&\quad = \delta_2^*\big(\lambda_{p+1}^*(s \otimes s \otimes W)\big)\\&\quad = \big(\lambda_p^* \otimes 1\big)(s \otimes s \otimes W) \quad \text{by diagram (2.2)}.\end{aligned}$$

Because $\ker\lambda_p^* \otimes 1 = \operatorname{im}\overline{\Delta}^{p-1} \otimes 1$, (b) follows.
The proof of part (a) is similar. $\square$

Definition 15. Let Ω be the span of $\{x^{\otimes^{i-1}} \otimes \mathcal{P}^m y \otimes x^{\otimes^{p-i+1}}\colon 1 \leqslant i \leqslant p+1\}$. Since $x, y, \mathcal{P}^m y$ are Borel generators, we may choose them to be part of a Borel basis for $H^*(X)$. We then may consider the components of an element in $\overline{H}^*(X)^{\otimes^{p+1}}$ in Ω.

Proposition 16. *$(1 \otimes \overline{\Delta}^{p-1})S_2$ and $-x \otimes \cdots \otimes x \otimes \mathcal{P}^m y + (\overline{\Delta}^{p-1} \otimes 1)S_1$ have the same components in Ω.*

Proof. $x^p \otimes W_1$ and $W_2 \otimes \beta_1 \mathcal{P}^m y$ have no component in Ω. Proposition 16 then follows from Proposition 14. □

Proof of Theorem 13. Let the component of $(1 \otimes \overline{\Delta}^{p-1})S_2$ in Ω be

$$\sum_{i=1}^{p+1} \gamma_i' x^{\otimes^{i-1}} \otimes \mathcal{P}^m y \otimes x^{\otimes^{p-i+1}}.$$

Let the component of $(\overline{\Delta}^{p-1} \otimes 1)S_1$ in Ω be

$$\sum_{i=1}^{p+1} \gamma_i x^{\otimes^{i-1}} \otimes \mathcal{P}^m y \otimes x^{\otimes^{p-i+1}}.$$

Note

$$\bar{x}^p = 0 \quad \text{implies } \gamma_1' = \gamma_{p+1} = 0 \tag{4.6}$$

because

$$\gamma_{p+1} = \langle \bar{x} \otimes \cdots \otimes \bar{x} \otimes \overline{\mathcal{P}^m y}, (\overline{\Delta}^{p-1} \otimes 1)S_1 \rangle = \langle \bar{x}^p \otimes \overline{\mathcal{P}^m y}, S_1 \rangle = 0,$$
$$\gamma_1' = \langle \overline{\mathcal{P}^m y} \otimes \bar{x} \otimes \cdots \otimes \bar{x}, (1 \otimes \overline{\Delta}^{p-1})S_2 \rangle = \langle \overline{\mathcal{P}^m y} \otimes \bar{x}^p, S_2 \rangle = 0.$$

By Proposition 16,

$$\gamma_{p+1}' = \langle \bar{x} \otimes \cdots \otimes \bar{x} \otimes \overline{\mathcal{P}^m y}, (1 \otimes \overline{\Delta}^{p-1})S_2 \rangle = -1 + \gamma_{p+1} = -1. \tag{4.7}$$

Now suppose $ad^{p-1}(x)(\overline{\mathcal{P}^m y}) = 0$. Then,

$$\begin{aligned} 0 &= \langle \bar{x} \otimes ad^{p-1}(\bar{x})(\overline{\mathcal{P}^m y}), S_2 \rangle = \langle \bar{x} \otimes Ad^{p-1}(\bar{x})(\overline{\mathcal{P}^m y}), (1 \otimes \overline{\Delta}^{p-1})S_2 \rangle \\ &= \sum_{i=2}^{p+1} \gamma_i'. \end{aligned} \tag{4.8}$$

Eqs. (4.6)–(4.8) imply

$$\sum_{i=1}^{p} \gamma_i' = 1. \tag{4.9}$$

Then

$$\begin{aligned} 0 &= \langle ad^{p-1}(\bar{x})(\overline{\mathcal{P}^m y}) \otimes \bar{x}, S_1 \rangle \\ &= \langle Ad^{p-1}(\bar{x})(\overline{\mathcal{P}^m y}) \otimes \bar{x}, (\overline{\Delta}^{p-1} \otimes 1)S_1 \rangle \\ &= \langle Ad^{p-1}(\bar{x})(\overline{\mathcal{P}^m y}) \otimes \bar{x}, (1 \otimes \overline{\Delta}^{p-1})S_2 \rangle \quad \text{by Proposition 16} \\ &= \sum_{i=1}^{p} \gamma_i'. \end{aligned}$$

This contradicts equation (4.9). Therefore $ad^{p-1}(\bar{x})(\overline{\mathcal{P}^m y}) \neq 0$. □

Appendix A

(1) E_6 and E_8 are both finite H-spaces. It is known [12] that if X is a finite H-space and $x \in PH_{2n}(X; \mathbb{F}_p)$ then $\bar{x}^p = 0$. So for $E_8 \times S^{2m+1} \times S^{2m+13} = X$ we have $\bar{x}_{12}^5 = 0$ and $\beta_1 \mathcal{P}^6 \zeta = 0$. Further, by [12] the only Steenrod operations that act nontrivially on $\bar{x}_{12}$ are $\bar{x}_{12}\beta_1$ and $\bar{x}_{12}\beta_1\mathcal{P}^1$.
Hence $(\bar{x}^{\otimes^{i-1}} \otimes \bar{y} \otimes \bar{x}^{\otimes^{5-i}})\beta_1\mathcal{P}^6 = 0$ and $(\bar{x}^{\otimes^{i-1}} \otimes \bar{y} \otimes \bar{x}^{\otimes^{6-i}})\beta_1\mathcal{P}^6 = 0$.
We conclude from the Main Theorem that if X is homotopy associative then $ad^l(\bar{x})(\bar{y}) \neq 0$ for $l \leqslant 3$, so there would have to be additional cohomology generators in degree $2m+25$, $2m+37$ dual to $ad^l(\bar{x})(\bar{y})$. Since m is not 1 or 5, such generators do not exist.

(2) $H^*(X;\mathbb{F}_3)\cong H^*(\widehat{E}_6;\mathbb{F}_3)\otimes\Lambda(y_3,\zeta_{11})$ with $\overline{\Delta}\zeta_{11}=x_8\otimes y_3$.
If $\beta_1\mathcal{P}^4\zeta_{11}\neq 0$ then $\mathcal{P}^4\zeta_{11}$ would be a primitive indecomposable in $H^{27}(X;\mathbb{F}_3)$. So it would produce an element of $QH^{27}(\widehat{E}_6;\mathbb{F}_3)$.
Consider the multiplicative fibration

$$\begin{tikzcd} K(\mathbb{Z}_3,8) \arrow[d, "j"] \\ \widehat{E}_6 \arrow[d, "\pi"] \\ E_6 \arrow[r] & K(\mathbb{Z}_3,9) \end{tikzcd} \tag{A.1}$$

By [11] such a 27-dimensional indecomposable maps by j^* to an indecomposable in $H^{27}(K(\mathbb{Z}_3,8))$. But $QH^{27}(K(\mathbb{Z}_3,8))=0$. Hence $\beta_1\mathcal{P}^4\zeta_{11}=0$.
Also by [12] the only nontrivial Steenrod operations on $\bar{x}_8$ are $\bar{x}_8\beta_1$, $\bar{x}_8\beta_1\mathcal{P}^1$. Hence

$$\big(\bar{x}^{\otimes^{i-1}}\otimes\bar{y}\otimes\bar{x}^{\otimes^{3-i}}\big)\beta_1\mathcal{P}^4=0,$$
$$\big(\bar{x}^{\otimes^{i-1}}\otimes\bar{y}\otimes\bar{x}^{\otimes^{3-i+1}}\big)\beta_1\mathcal{P}^4=0.$$

To check if $\bar{x}_8^3=0$, if $\bar{x}_8^3\neq 0$, there exists an indecomposable in $H^{24}(X;\mathbb{F}_3)$. By [10] such a generator is indecomposable in $H^{24}(\widehat{E}_6;\mathbb{F}_3)$ and restricts to a generator or a cube in $H^*(K(\mathbb{Z}_3,8);\mathbb{F}_3)$. The only possibilities are $\mathcal{P}^1\mathcal{P}^3 i_8$ or $\mathcal{P}^3\mathcal{P}^1 i_8$. But $\mathcal{P}^3x_9=0=\mathcal{P}^1x_9$ so such generators lie in the image of $\mathcal{P}^1$ or $\mathcal{P}^3$. Hence

$$\big\langle\bar{x}_8^3,\mathcal{P}^1v_1\big\rangle\neq 0\quad\text{or}\quad\big\langle\bar{x}_8^3,\mathcal{P}^3v_2\big\rangle\neq 0.$$

However $(\bar{x}_8^3)\mathcal{P}^1=0$, $(\bar{x}_8^3)\mathcal{P}^3=0$, so this is impossible and $\bar{x}_8^3=0$. By the Main Theorem if X were homotopy associative $ad^2(\bar{x}_8)(\bar{y})\neq 0$ or there would be an additional generator in degree 19. If we examine the Serre spectral sequence for the fibration there are no 19-dimensional generators in $H^*(E_6)$ or $H^*(K(\mathbb{Z}_3,8))$ and the first odd generator in $E_\infty^{p,q}$ for $p>0$, $q>0$ occurs in degree 25. Hence $QH^{19}(X;\mathbb{F}_3)=0$ and X cannot be homotopy associative.

(3) The element x_{23} lies in $\ker\beta_1\mathcal{P}^{11}$ and $\overline{\Delta}x_{23}=x_{12}\otimes x_{11}$ [4,13]. Further, $\bar{x}_{12}^5=0$ by [12] and $\langle\bar{x}_{12},\beta_1\mathcal{P}^5x_{11}\rangle=0$. Theorem 13 with $\zeta=x_{23}$ implies $ad^4(\bar{x}_{12})(\bar{x}_{51})\neq 0$.

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Further reading

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